Knowledge Compilation: Theory, Practice, and Applications

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Outline

How to model using SAT

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The SAT problem

$$\neg a \vee b \qquad \begin{array}{l} a \vee \neg b \\ \neg b \vee c \vee d \\ e \vee f \end{array} \qquad \begin{array}{l} \neg a \vee c \\ \neg b \vee c \vee \neg d \\ e \vee \neg f \end{array} \qquad \neg b \vee c$$

- Propositional variables: a, b, c, d, e, f
- Literals: $a, \neg a$
- Clauses: $a \vee b$
- A `CNF` formula F: $(a \vee \neg b) \wedge (\neg a \vee c) \wedge \ldots$

The SAT problem

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a	b	c	d	e	f

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- An assignment is a mapping $\mathscr{I} : var(F) \rightarrow \{\bot, \top\}$ and

  $\mathscr{I}$ is satisfying if results in F being $\top$ (true)

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- SAT problem: does there exist a satisfying assignment for F?

The SAT problem

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a	b	c	d	e	f
$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$

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a	b	c	d	e	f
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$

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A Brief Detour into Complexity Theory

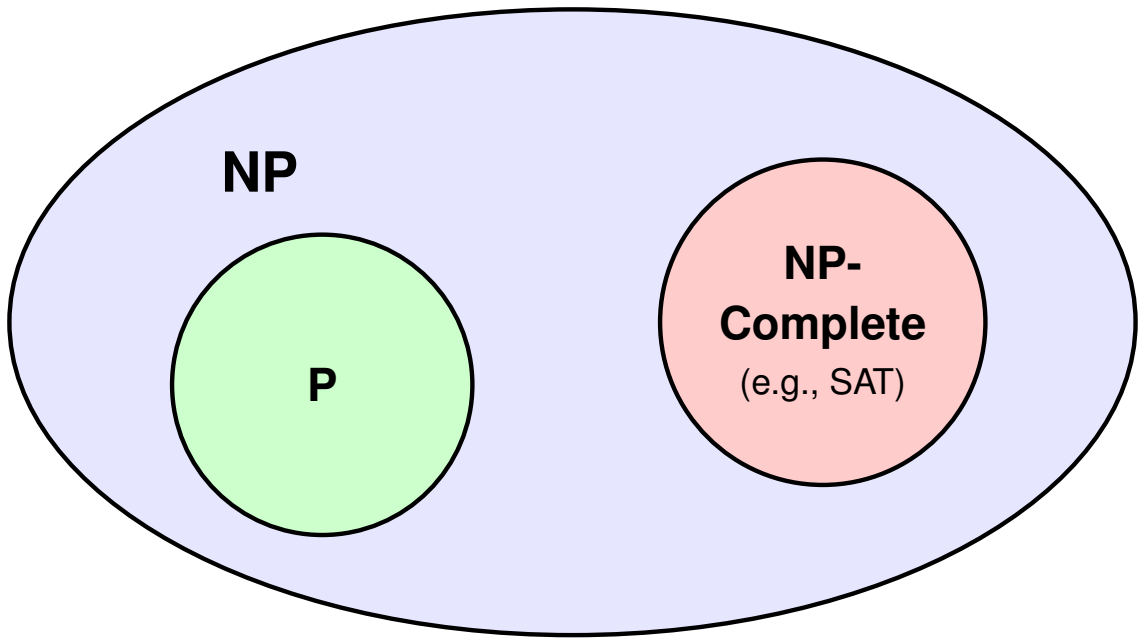
- To understand why using SAT, we must understand how hard SAT is to solve.
- **Class P (Polynomial Time):**
  - Problems that can be solved efficiently by a deterministic algorithm.
  - Examples: Sorting an array, finding the shortest path in a graph.
- **Class NP (Nondeterministic Polynomial Time):**
  - Problems where a proposed solution can be verified efficiently.
  - Example: Sudoku. Hard to solve from scratch, but easy to check if a completed grid is correct.

Does P = NP? (2nd Millennium Prize Problem of Mathematics)

SAT and NP-Completeness

- **Cook-Levin Theorem (Cook, 1971):** SAT was the first problem proven to be NP-Complete.
- What does NP-Complete mean?
  - It is in NP (a satisfying assignment is easy to check).
  - It is as hard as the hardest problems in NP.
  - *Every* problem in NP can be translated (reduced) into a SAT problem efficiently.

- **ETH (Exponential-Time Hypothesis):** Worst-case SAT cannot be solved significantly better than exponential time $\mathcal{O}(2^n)$ (more precisely: there is no subexponential-time algorithm for 3-SAT).



The SAT Solving Paradox

Theory vs. Practice

- **Theory:** SAT is NP-Complete ⇒ intractable in the worst case.
- **Practice:** Modern SAT solvers can routinely solve industrial instances with millions of variables and clauses.

- **How is this possible?**
  - Real-world problems (like hardware verification or XAI) are highly structured, unlike random formulas.
  - Modern solvers exploit this hidden structure.

- **The CDCL (Conflict-Driven Clause Learning) Revolution (Silva and Sakallah, 1996):**
  - When the solver makes a mistake, it analyzes the logical conflict.
  - It *learns* a new clause to prevent making the exact same mistake again.
  - It then backtracks non-chronologically to jump out of dead search spaces.

Outline

Graph Coloring Problem

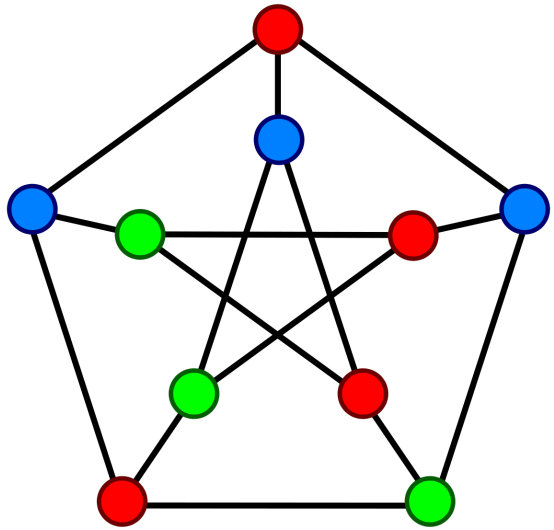
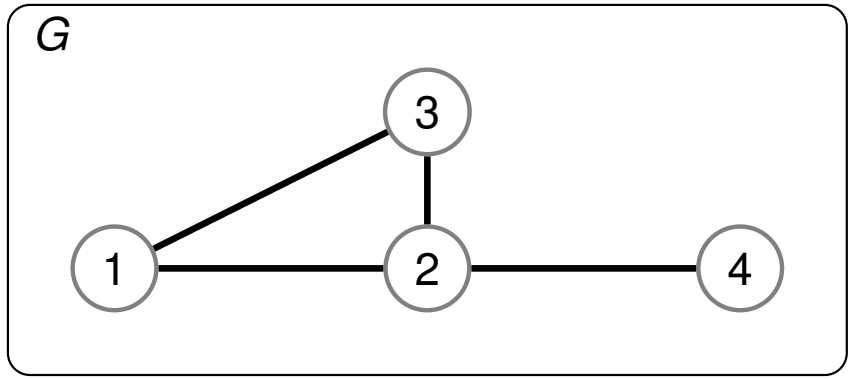


Figure: $\chi(G) = 3$

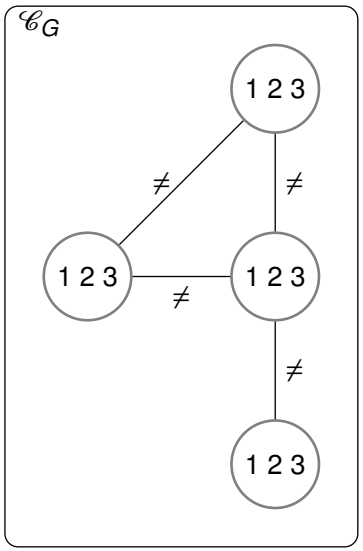
- To assign a minimum number of colors to all vertices s.t. no adjacent vertices have the same color
- The smallest number of colors: *chromatic number* ($\chi(G)$)
- Determining the chromatic number of a graph is an NP-hard task

Classical Graph Coloring Encoding

- *Decision Problem*: Can we color $G = (V, E)$ with 3 colors?



- We can express the problem with a set of constraints



1. x_{ij} is true if the vertex i is colored with the color j
2. At least one color by vertex: $\bigwedge_{i=1}^{4}(\bigvee_{j=1}^{3} x_{ij})$
3. At most one color by vertex: $\bigwedge_{i=1}^{4}\bigwedge_{j=1}^{3}\bigvee_{k=j+1}^{3}(\neg x_{ij} \vee \neg x_{ik})$
4. Two adjency vertices cannot share the same color: $\forall \{i,j\} \in E, \bigwedge_{k=1}^{3} \neg x_{ik} \vee \neg x_{jk}$

Live example?

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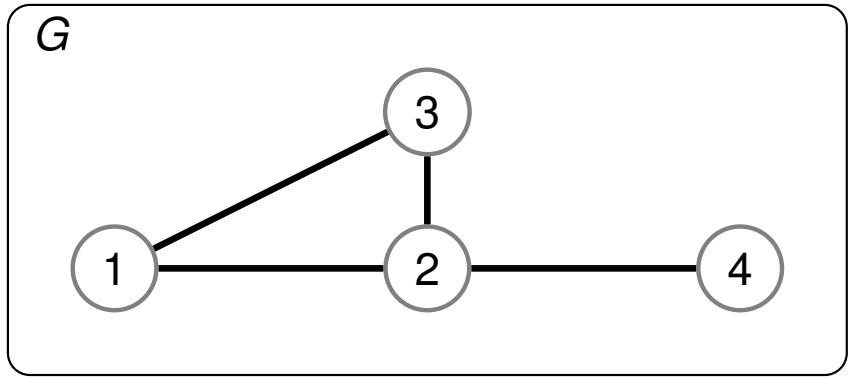

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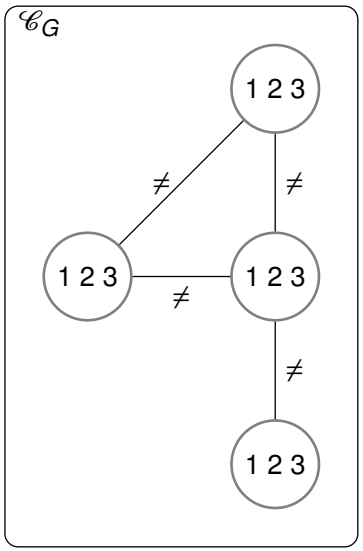

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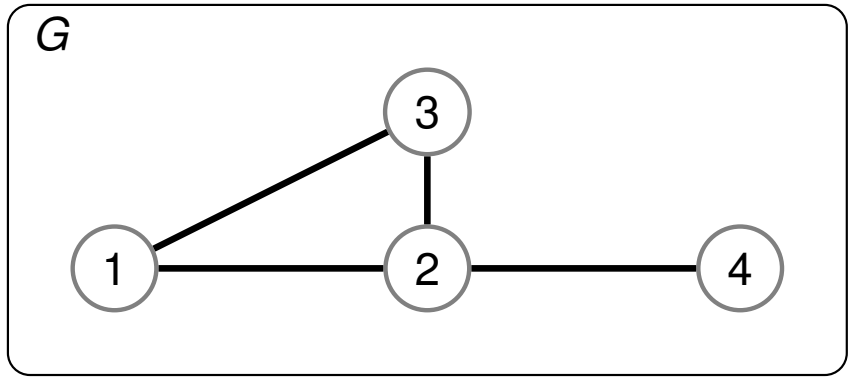

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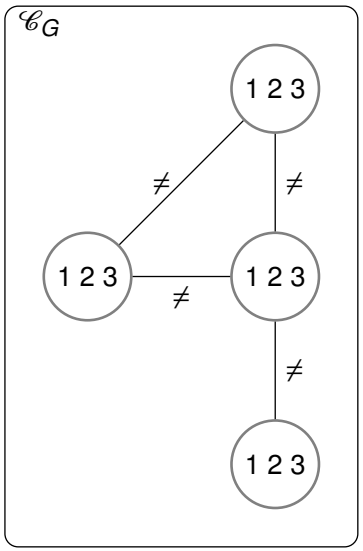

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Live example?

The Counting Challenge

Finding one coloring is “easy” (SAT). But how many colorings exist?

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The Model Counting Problem (#SAT)

Definition

Model Counting is the problem of computing the number of total satisfying assignments (models) for a given propositional formula Σ.

$$\Sigma \mapsto \|\Sigma\| = |\{\mathscr{I} \mid \mathscr{I} \models \Sigma\}|$$

Example: Let $\Sigma = (\neg a \vee \neg b \vee \neg c) \wedge (a \vee c) \wedge (a \vee b) \wedge (\neg b \vee \neg c)$

The models of Σ over variables $\{a, b, c\}$ are:

a	b	c
$\top$	$\bot$	$\bot$
$\top$	$\bot$	$\top$
$\top$	$\top$	$\bot$

Answer: $\|\Sigma\| = 3$

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The Complexity of Model Counting

- **A Central Problem in AI**
  Model counting is crucial for various applications:
  - Probabilistic reasoning (Bayesian inference)
  - Automated verification
  - Planning under uncertainty

- **Computational Hardness**
  #SAT is **#P-complete** (Valiant, 1979).
  - It is significantly harder than SAT (NP-complete).

The Power of Counting: Toda's Theorem 1991

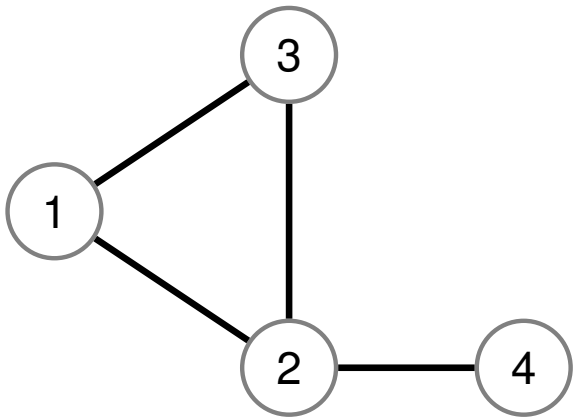
Seinosuke Toda (Gödel Prize 1998)

The entire Polynomial Hierarchy is contained in $P^{\#P}$:

$$PH \subseteq P^{\#P}$$

Outline

The SAT Encoding in Practice



$G = (V, E)$

CNF Formula Σ_G ($n = 4, k = 3$) where $X = \text{var}(\Sigma_G)$

1. At least one color per vertex:

$$(x_{1,1} \vee x_{1,2} \vee x_{1,3}) \wedge (x_{2,1} \vee x_{2,2} \vee x_{2,3}) \wedge (x_{3,1} \vee x_{3,2} \vee x_{3,3}) \wedge (x_{4,1} \vee x_{4,2} \vee x_{4,3})$$

2. At most one color per vertex (Mutual Exclusion):

$$(\neg x_{1,1} \vee \neg x_{1,2}) \wedge (\neg x_{1,1} \vee \neg x_{1,3}) \wedge (\neg x_{1,2} \vee \neg x_{1,3})$$
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3. Adjacency (Edge Constraints):

$$(\neg x_{1,1} \vee \neg x_{2,1}) \wedge (\neg x_{1,2} \vee \neg x_{2,2}) \wedge (\neg x_{1,3} \vee \neg x_{2,3}) \quad \textit{(Edge 1–2)}$$
$$(\neg x_{1,1} \vee \neg x_{3,1}) \wedge (\neg x_{1,2} \vee \neg x_{3,2}) \wedge (\neg x_{1,3} \vee \neg x_{3,3}) \quad \textit{(Edge 1–3)}$$
$$(\neg x_{2,1} \vee \neg x_{3,1}) \wedge (\neg x_{2,2} \vee \neg x_{3,2}) \wedge (\neg x_{2,3} \vee \neg x_{3,3}) \quad \textit{(Edge 2–3)}$$
$$(\neg x_{2,1} \vee \neg x_{4,1}) \wedge (\neg x_{2,2} \vee \neg x_{4,2}) \wedge (\neg x_{2,3} \vee \neg x_{4,3}) \quad \textit{(Edge 2–4)}$$

- **Variables:** $4 \times 3 = 12$ Boolean variables.
- **Clauses:** $4 + (4 \times 3) + (4 \times 3) = 28$ clauses.
- **The Goal:** Compute $\|\Sigma_G\| = |\{\mathscr{I} \in \{0,1\}^X \mid \mathscr{I} \models \Sigma_G\}|$.

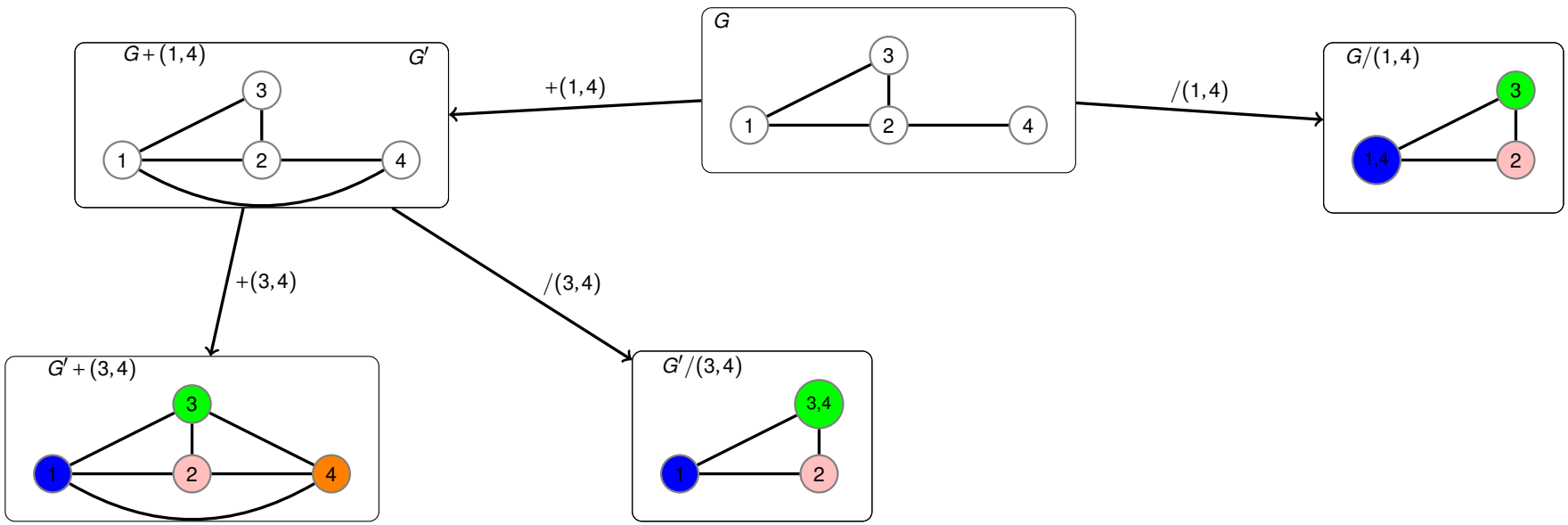
Even for this small graph, a manual count is tedious. Knowledge Compilation automates this by transforming Σ_G into a tractable form.

Zykov's deletion-contraction recurrence

Operation on the graph:

- We branch on the missing edges;
- G+(i,j) means G with the edge (i,j);
- G/(i,j) means we merge the vertices i and j in G.

Zykov's recurrence: $\chi(G) = \min\{\, \chi(G/(ij)),\ \chi(G+(ij)) \,\}\ \forall (i,j) \notin E$



Encoding Zykov's Graph Coloring in CNF (Glorian et al., 2019)

1. The Equivalence Relation (s_{ij})

$s_{ij} \equiv$ vertices i and j share the same color.

- **Edge Constraints:** $\neg s_{ij} \quad \forall (i,j) \in E$
- **Transitivity:** $(\neg s_{ij} \vee \neg s_{jk} \vee s_{ik}) \quad \forall i,j,k \in V$

2. The k-Color Constraint

Identify "color leaders" c_j who start a new color:

$$c_j \iff \bigwedge_{i<j} \neg s_{ij} \quad \text{and ensure} \quad \sum_{j=1}^{n} c_j \leq k$$

The Compilation Challenge

Encoding $\sum c_j \leq k$ into CNF requires **auxiliary variables** (e.g., via Totalizers or Adder circuits).

- We only care about the models over the original s_{ij} and c_j variables.
- **Problem:** Standard counting over the compiled circuit will be biased by these extra bits.

Outline

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Projected Model Counting (∃#SAT)

The Problem: Auxiliary Variables

Modern SAT encodings Σ often use a set of variables $\mathbf{X} = \mathbf{P} \cup \mathbf{A}$:

- $\mathbf{P}$: **Primary variables** (e.g., the actual colorings x_{ik}).
- $\mathbf{A}$: **Auxiliary variables** (e.g., Tseytin variables, cardinality counters).

Definition (Projected Model Count)

Given a formula Σ over $\mathbf{P} \cup \mathbf{A}$, the projected model count over $\mathbf{P}$ is the number of assignments to $\mathbf{P}$ that can be extended to a model of Σ:

$$\|\Sigma\|_{|\mathbf{P}} = \left|\{\mathscr{I}_{\mathbf{P}} \mid \exists \mathscr{I}_{\mathbf{A}} : (\mathscr{I}_{\mathbf{P}} \cup \mathscr{I}_{\mathbf{A}}) \models \Sigma\}\right|$$

Standard #SAT
Counts every bit-string
$|\text{Models}| = 2^{|\mathbf{A}|} \times |\text{Solutions}|$
(if $\mathbf{A}$ is free/unconstrained)

$\Rightarrow$

Projected #SAT
Counts unique solutions
Ignores the "internal logic" of the encoding.

*Crucial for Knowledge Compilation: Can we compile a circuit that supports **Existential Quantification** efficiently?*

Outline

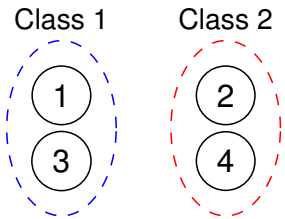
Revisiting Zykov: Partitions vs. Colorings

What are we actually counting?

Recall the Zykov variables s_{ij} (true if i, j share a color). A satisfying assignment of Σ_{Zykov} represents a **partition** of V into $m \leq k$ non-empty sets (equivalence classes).

- **The Discrepancy:** One partition into m color classes corresponds to multiple ways to assign k actual colors.

One Partition ($m = 2$)



Ways to color with $k = 3$:
Pick 2 colors out of 3, then arrange:

$$P(3,2) = \frac{3!}{(3-2)!} = \textbf{6} \text{ colorings}$$

The Problem

If a partition uses m color classes, it represents $k!/(k-m)!$ colorings. A standard #SAT count on Σ_{Zykov} treats every partition as 1, regardless of m.

Solution: Assign a **weight** to each model based on the number of color classes m.

Outline

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Weighted Model Counting

Weighted CNF Formula

- Let $V = \{v_1, \ldots, v_n\}$ be a set of Boolean variables.
- Let $L = \{v_i, \overline{v}_i \mid v_i \in V\}$ be the set of literals over V.
- A wCNF formula over V is a pair (F, w) such that:
  - F is a conjunction of clauses,
  - w is a map from L to $\mathbb{Q}$ (i.e., a weighting of literals)

Weighted Model Counting

- The weight of a variable assignment $\boldsymbol{v} \in \{0,1\}^n$ is the product of weights of literals which are true in $\boldsymbol{v}$:

$$W(\boldsymbol{v}) = \prod_{\ell \in L, \boldsymbol{v} \models \ell} w(\ell)$$

- The weighted model count of a wCNF formula is the sum of weights of the models of F:

$$W(F) = \sum_{\boldsymbol{v} \in \{0,1\}^n, \boldsymbol{v} \models F} W(\boldsymbol{v})$$

Outline

Questions

Qualitative (Decision or Optimization)

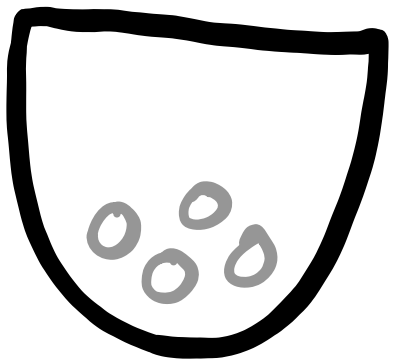
- Does the problem have a (optimal/preferred) solution?
- Output a reason for a decision.

Quantitative Questions

- How many solutions does the problem have?
- How likely is an occurrence?

Bayesian Inference as Counting Problem

Example



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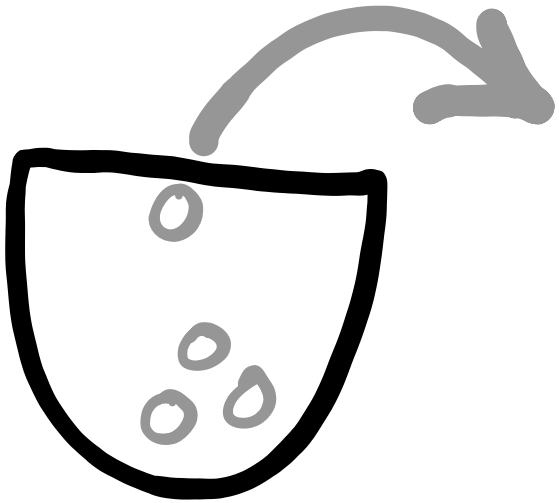

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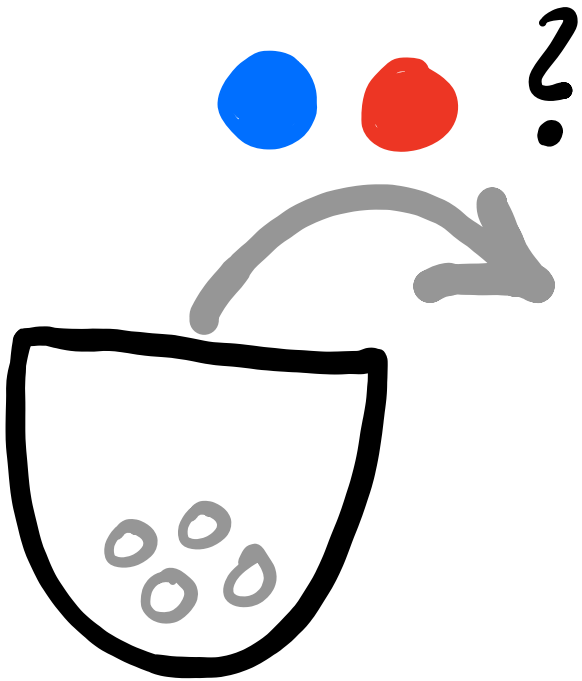

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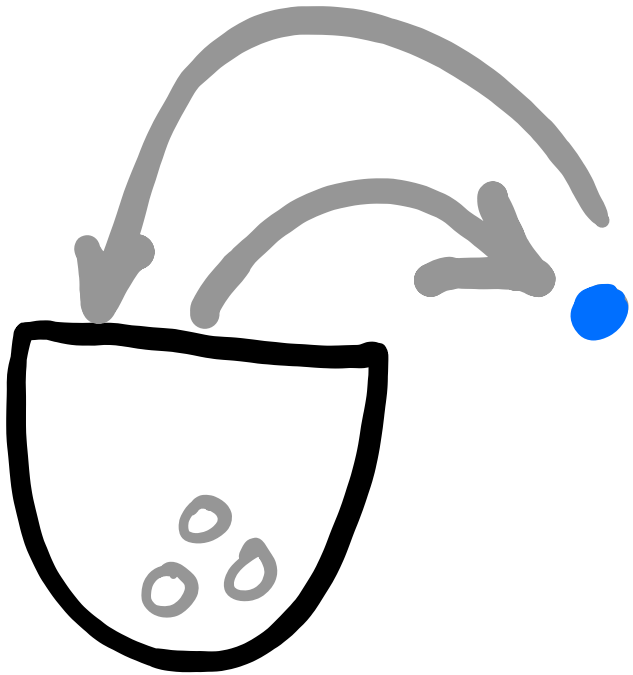

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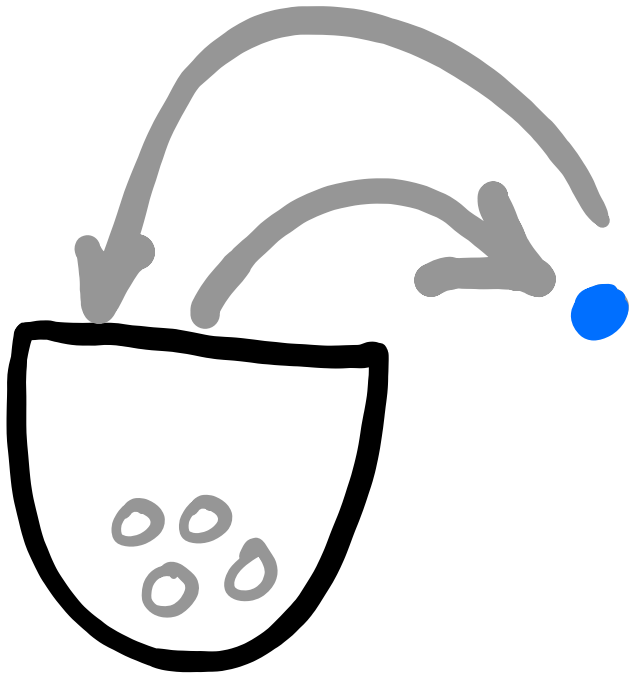

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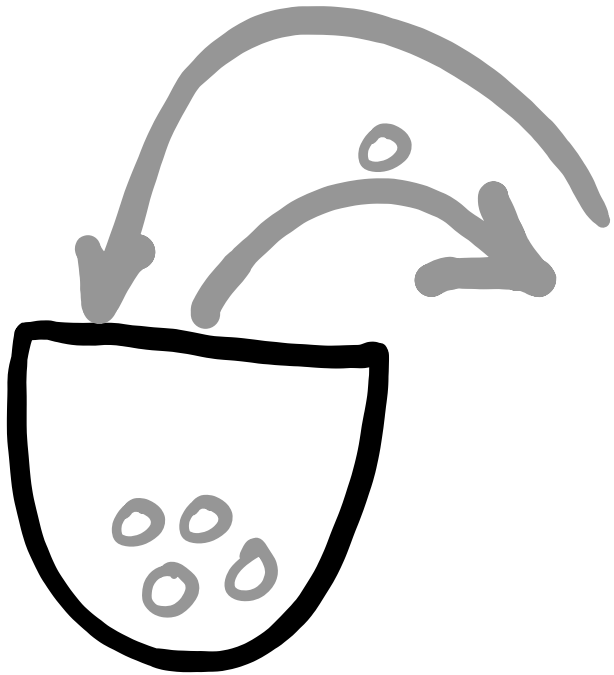

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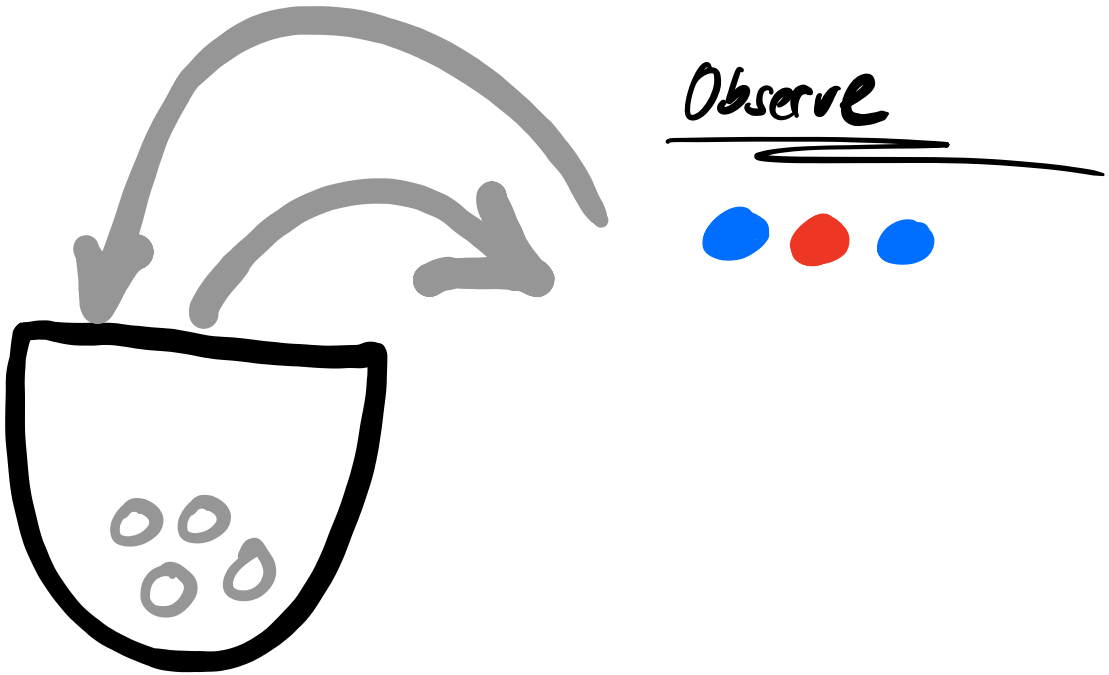

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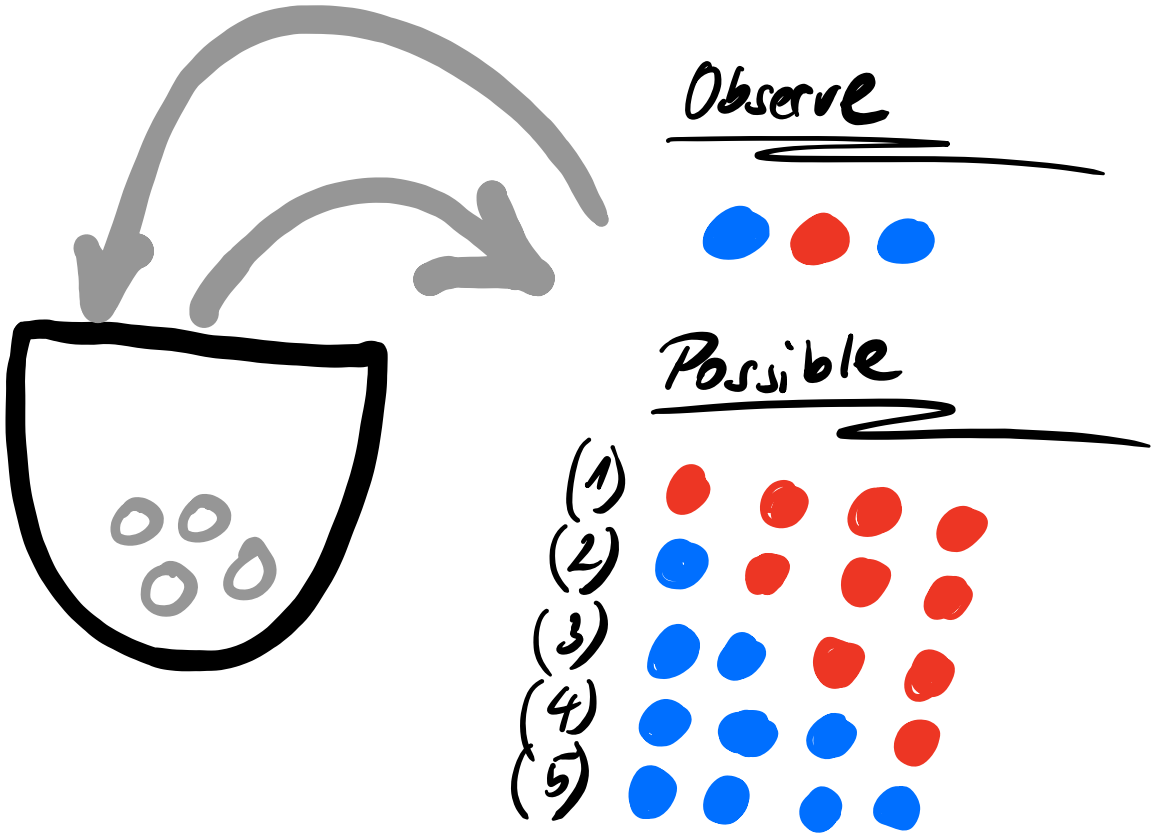

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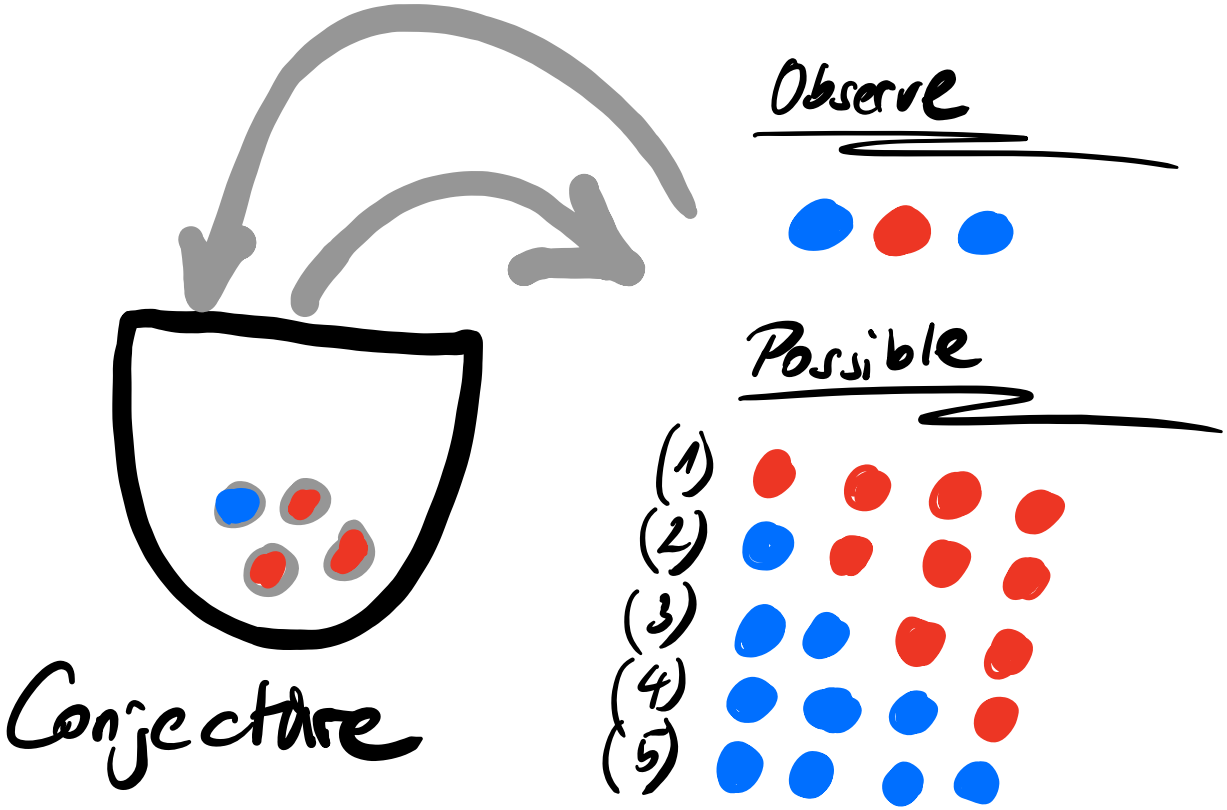

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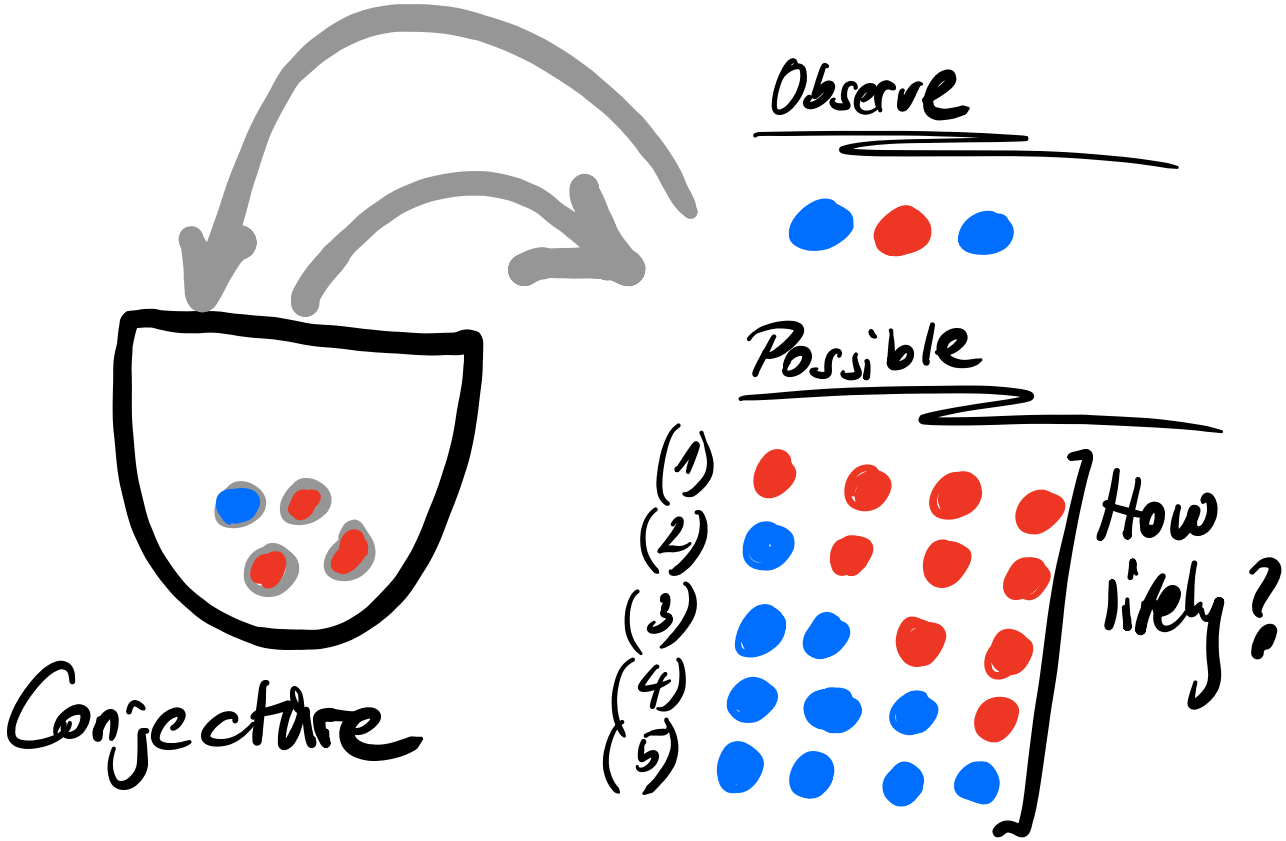

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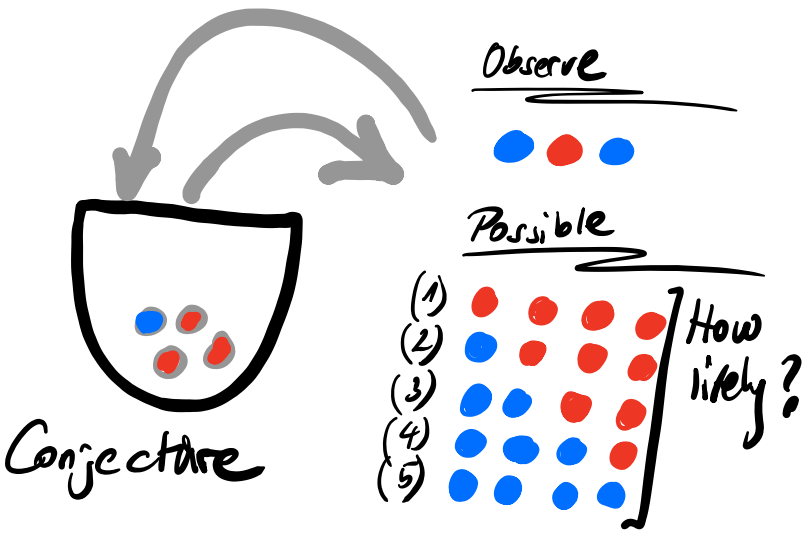


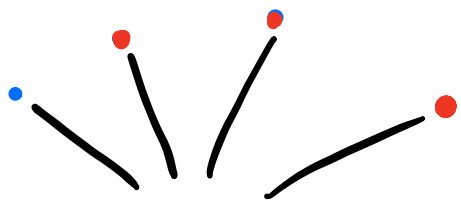
- How “likely”

⇒ Notion of Probability

- How does this relate to counting?

Bayesian Inference as Counting Problem

What possible combinations can we obtain?



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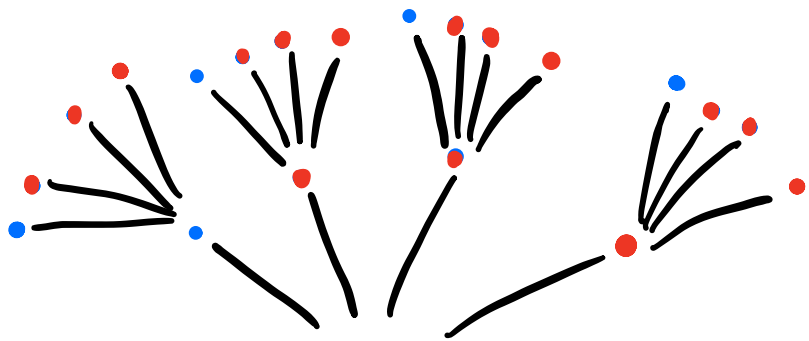

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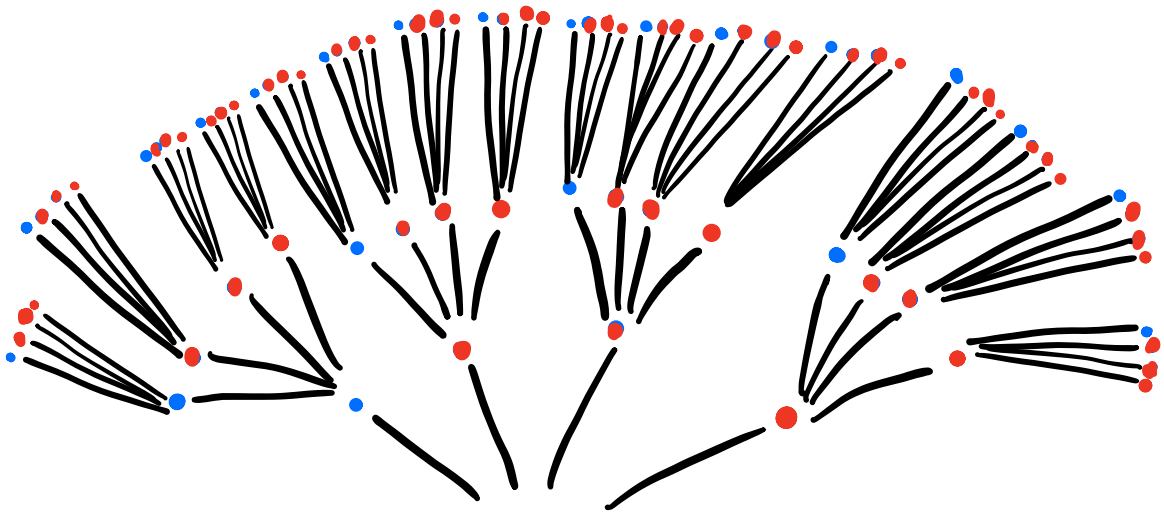

Bayesian Inference as Counting Problem

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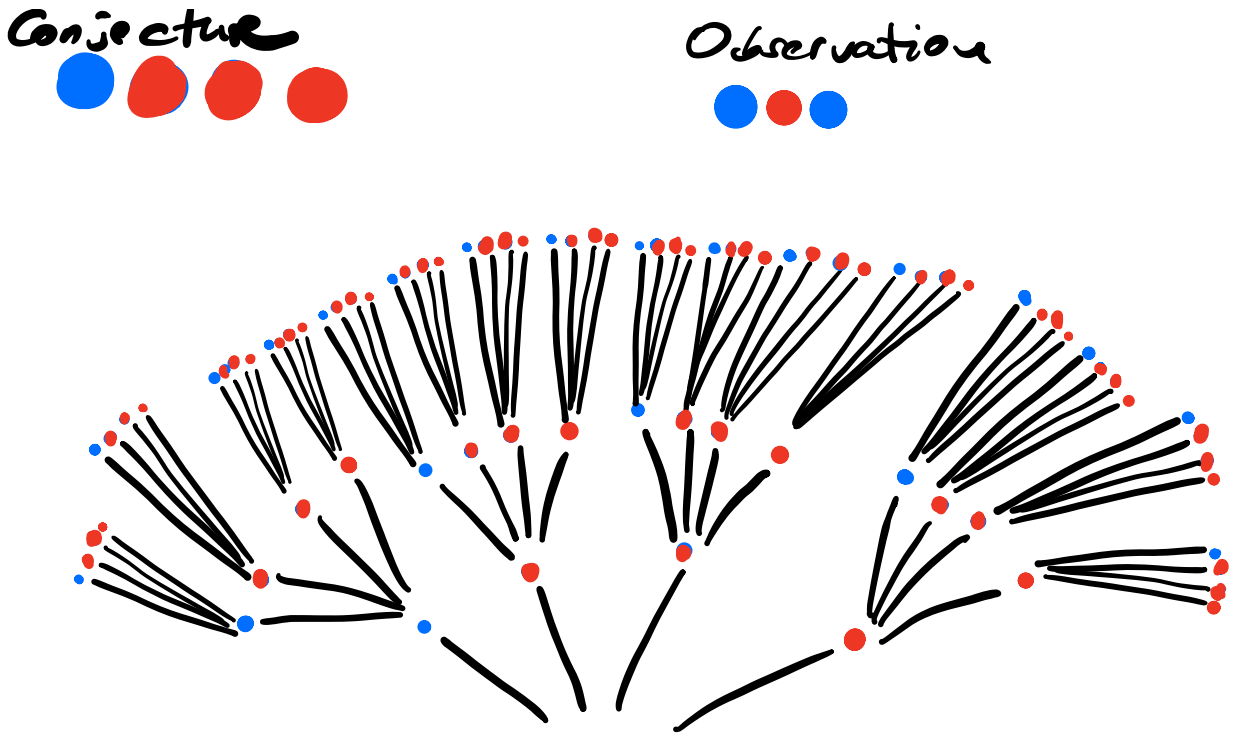

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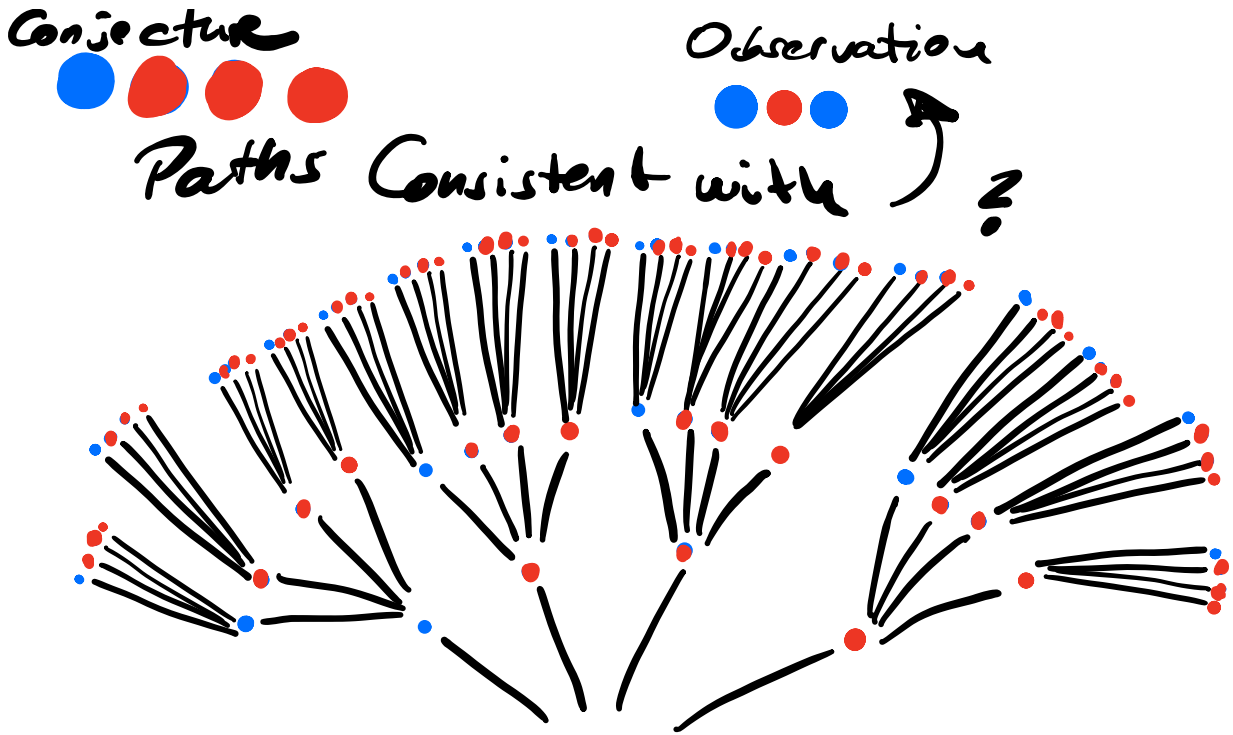

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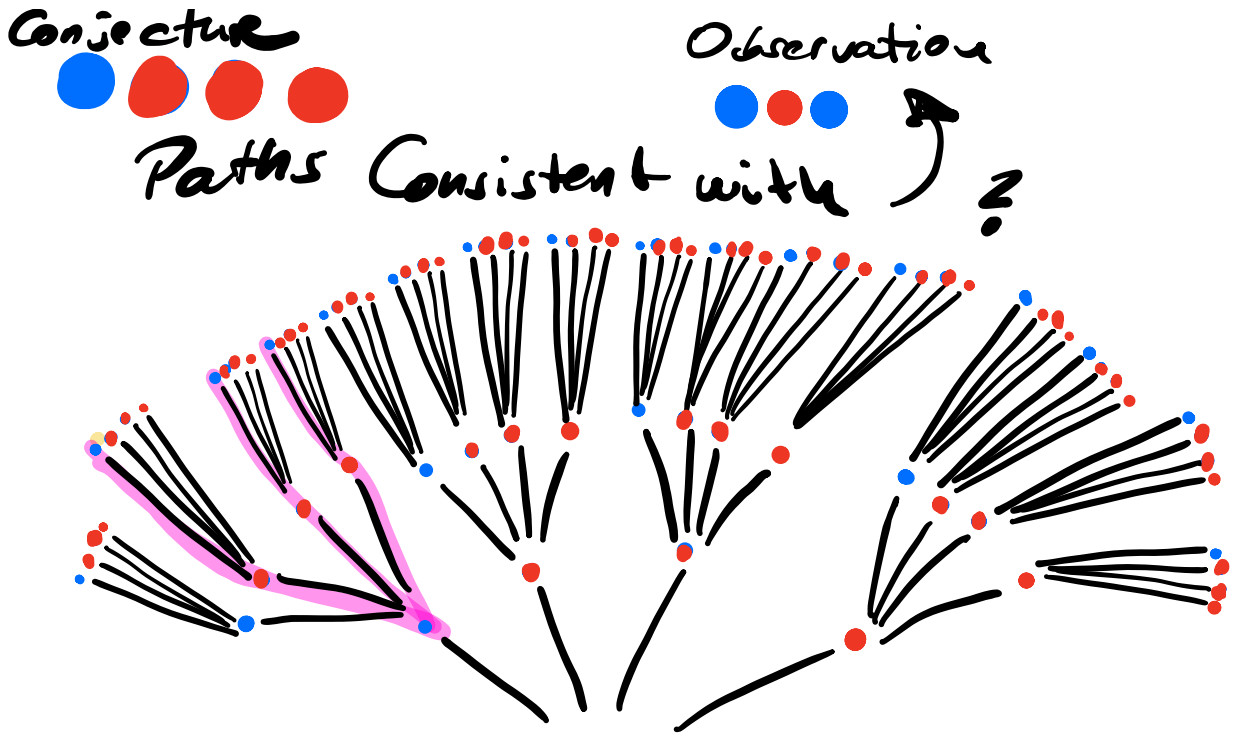

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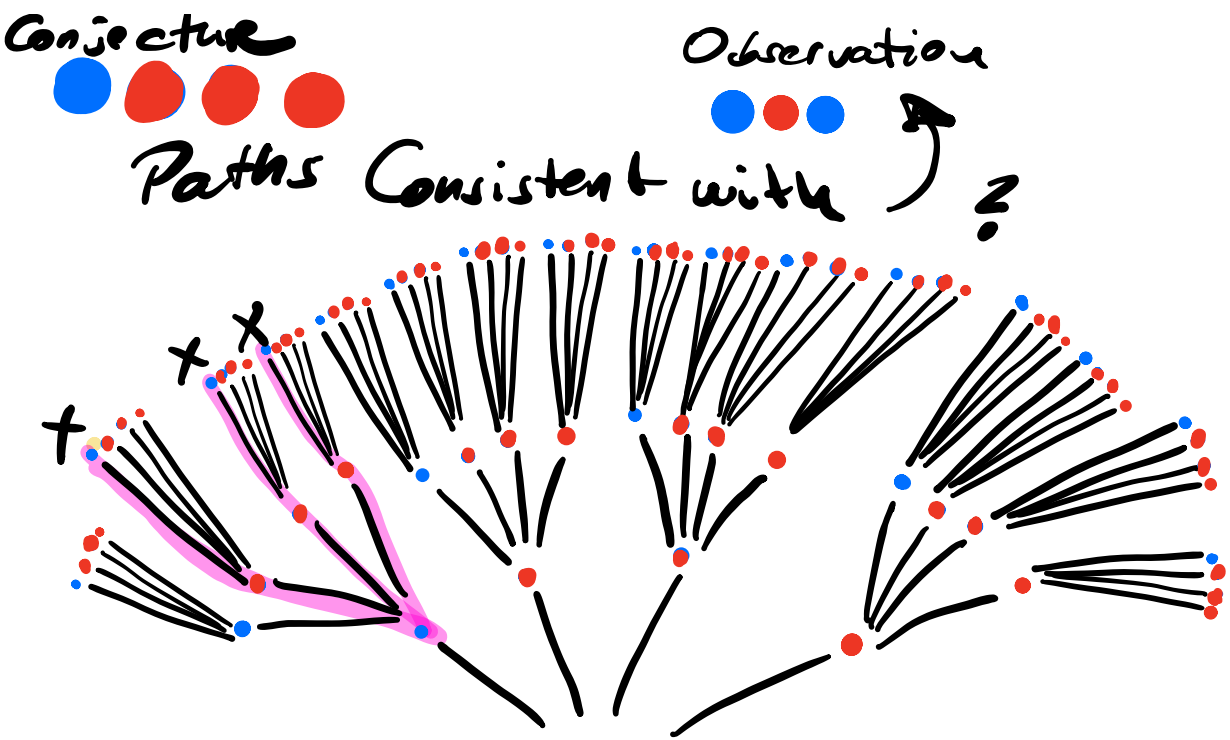

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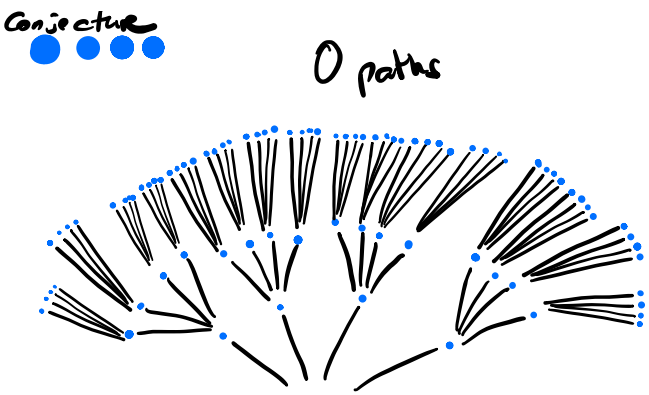

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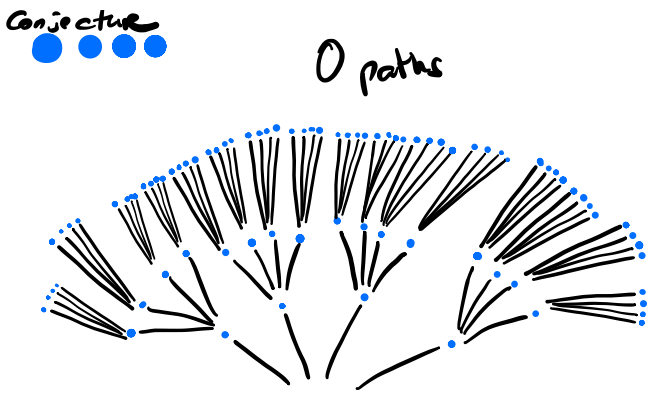

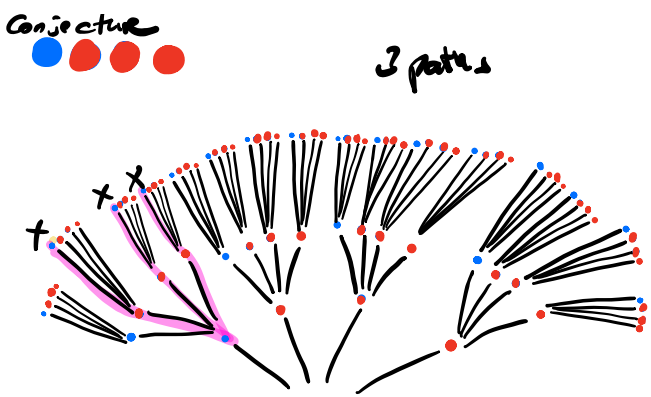

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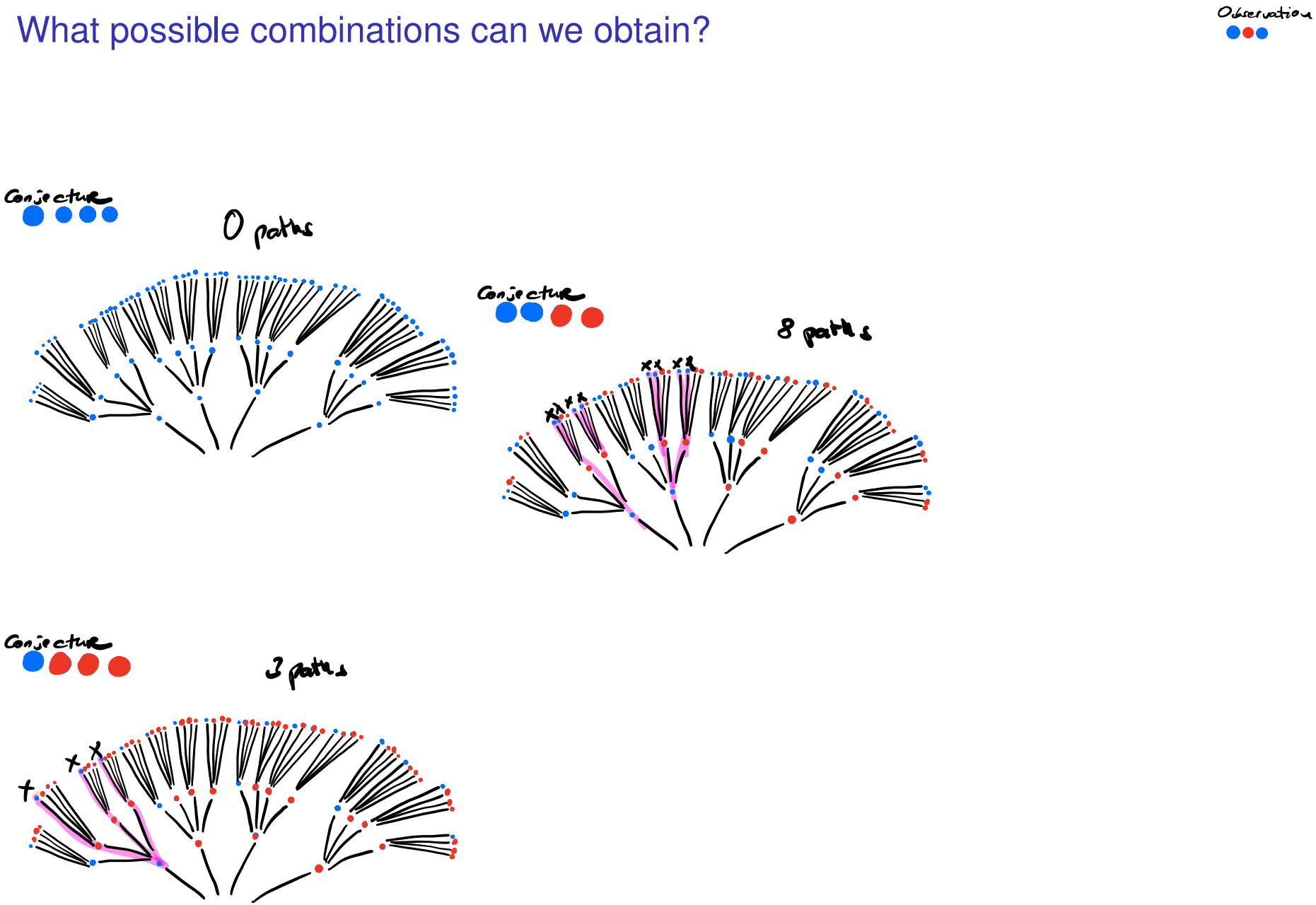

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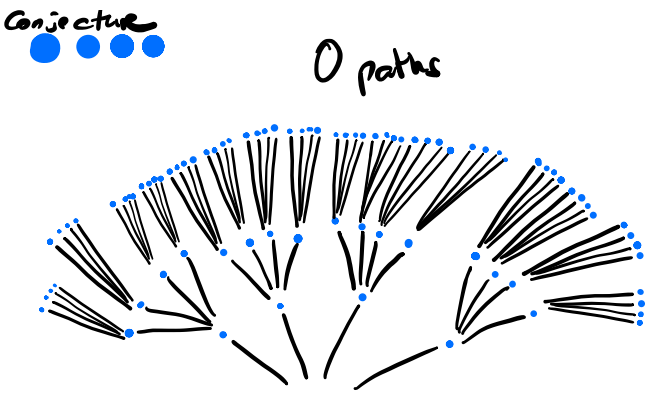

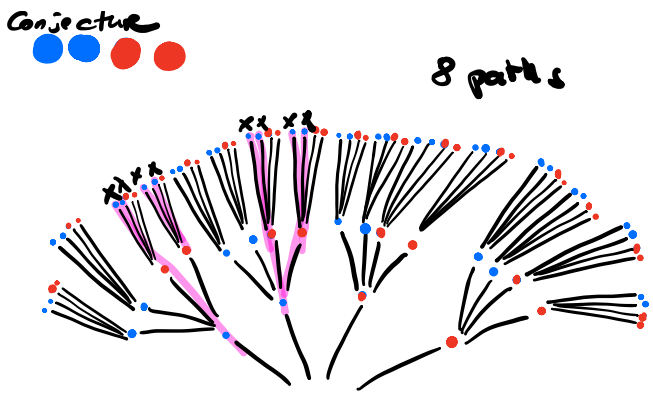

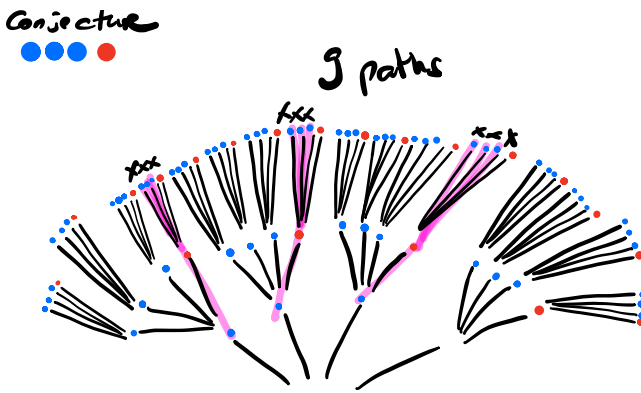

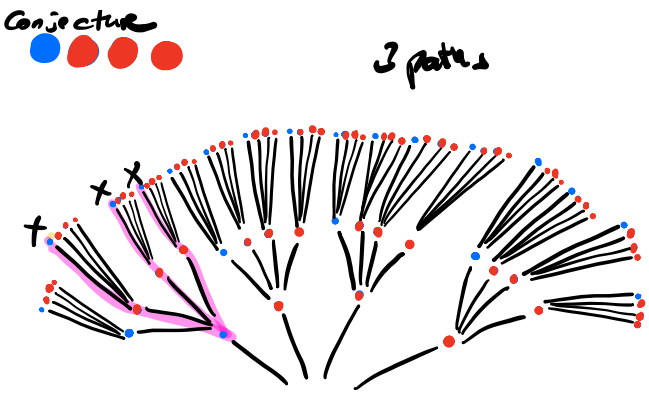

Bayesian Inference as Counting Problem

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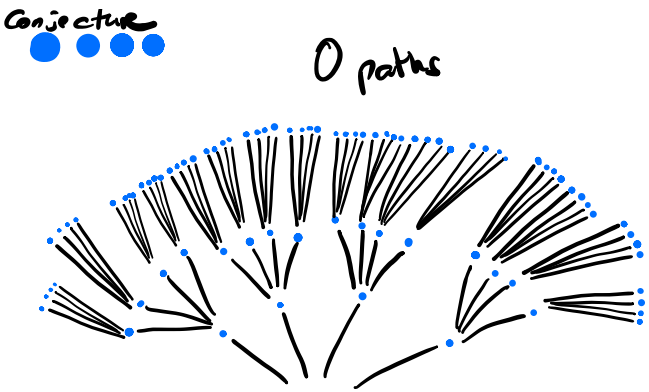
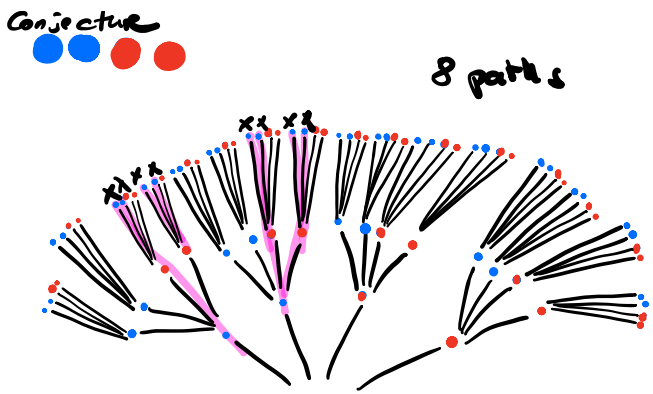
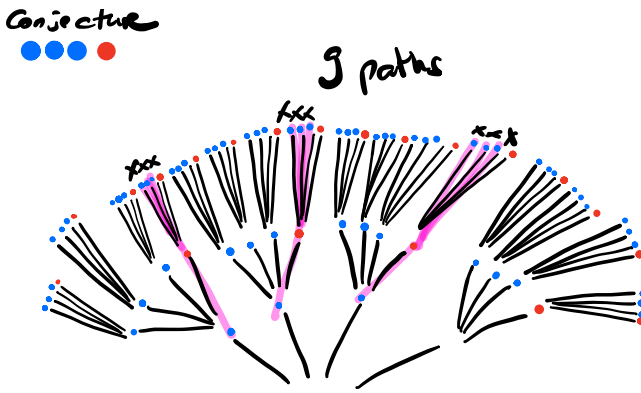
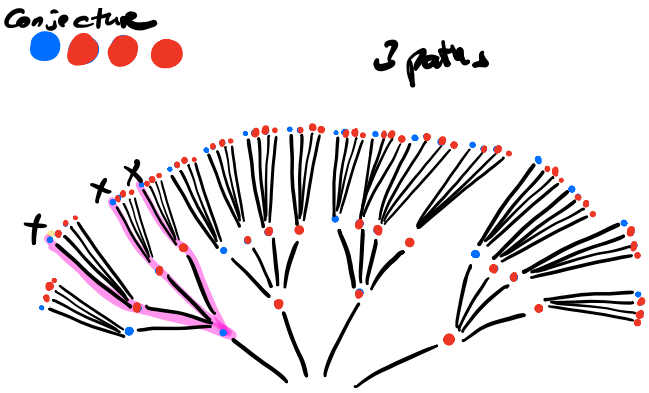
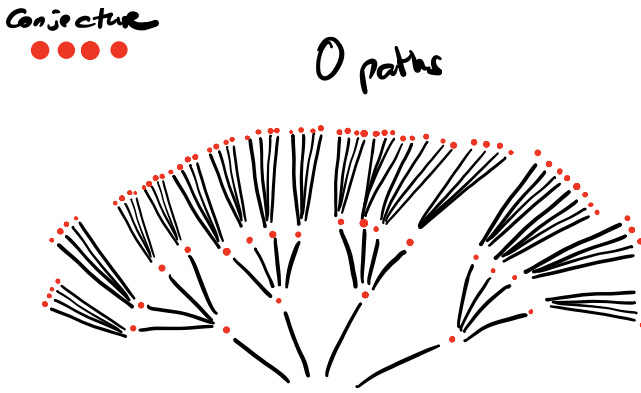










Bayesian Inference as Counting Problem

What possible combinations can we obtain?

Conjecture	Possible Paths for ⟨●●●⟩
[●●●●]	0
[●●●●]	9
[●●●●]	8
[●●●●]	3
[●●●●]	0
Total:	20

Bayesian Inference as Counting Problem

What possible combinations can we obtain?

Conjecture	Possible Paths for ⟨●●●⟩
[●●●●]	0
[●●●●]	9
[●●●●]	8
[●●●●]	3
[●●●●]	0
Total:	20

Probability: relationship between actual and possible occurrence

Bayesian Inference as Counting Problem

What possible combinations can we obtain?

Conjecture	Possible Paths for ⟨●●●⟩
[●●●●]	0
[●●●○]	9
[●●○○]	8
[●○○○]	3
[○○○○]	0
Total:	20

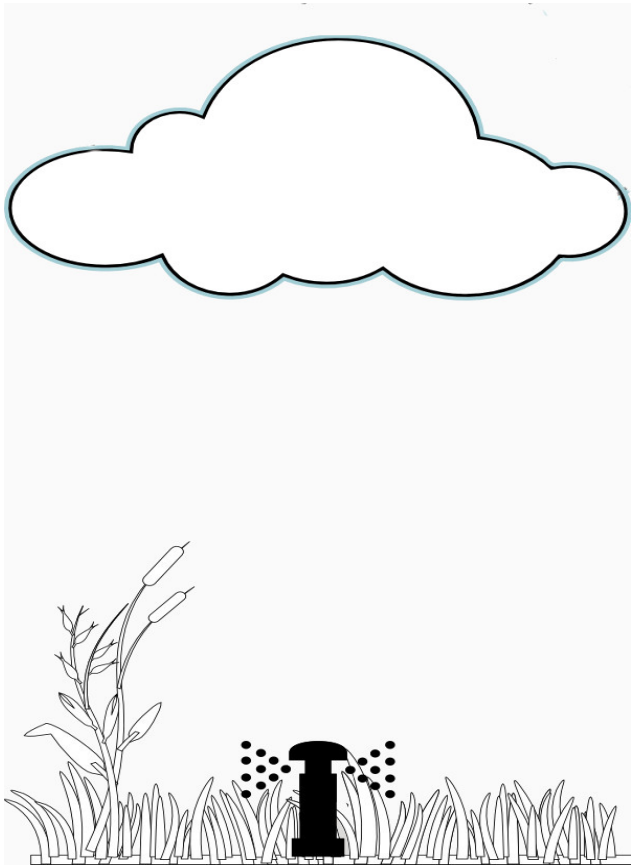
Probability [●○○○] "=" $\frac{3}{0+9+8+3+0} = \frac{3}{20}$

Bayesian Inference as Counting Problem

What possible combinations can we obtain?

Conjecture	Possible Paths for ⟨●●●⟩	Probability
[●●●●]	0	$0/20 = 0$
[●●●●]	9	$9/20 = 0.45$
[●●●●]	8	$8/20 = 0.40$
[●●●●]	3	$3/20 = 0.15$
[●●●●]	0	$0/20 = 0$
Total:	20	1

Probabilistic Inference



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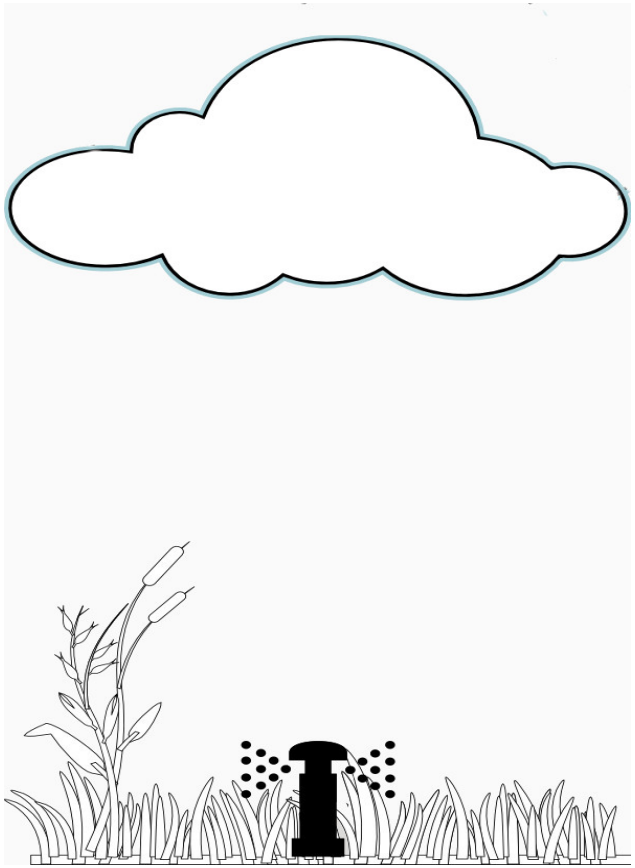
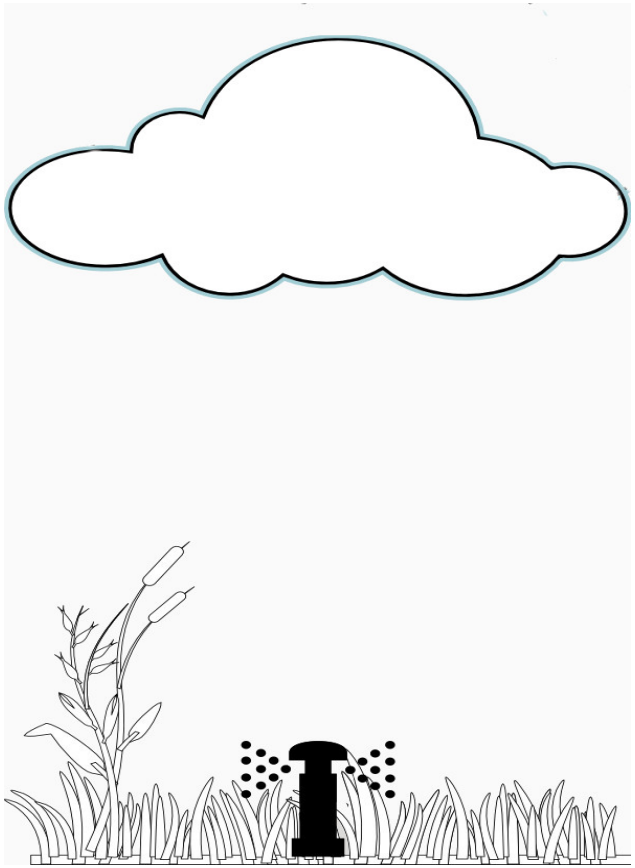


- Observations:
  - It is cloudy
  - The grass is wet
  - ...

Probabilistic Inference

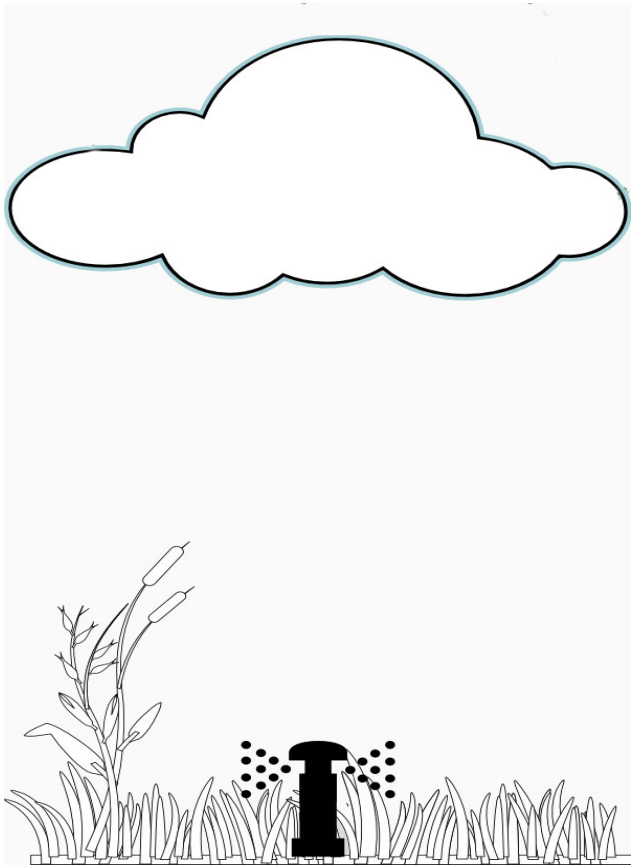


- Observations:
  - It is cloudy
  - The grass is wet
  - ...

day	cloudy	wet grass
1	1	1
2	1	1
3	1	0
4	0	0

$$\mathbb{P}(\text{wet grass}|\text{cloudy}) = \frac{2}{3}$$

Probabilistic Inference



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day	cloudy	wet grass
1	1	1
2	1	1
3	1	0
4	0	0

$$\mathbb{P}(\text{wet grass}|\text{cloudy}) = \frac{2}{3}$$

- Modeling a problem using a joint distribution on a set of random variables

Probabilistic Inference

- Let $\mathbf{X}$ be a set of random variables over finite domains, here mostly $\{0, 1\}$
- $\mathrm{Inst}(\mathbf{X})$ is the set of all complete assignments to $\mathbf{X}$
- An instantiation $\mathbf{x} \in \mathrm{Inst}(\mathbf{X})$ assigns to each variable $X_i \in X$ a value (from its domain)
- A probability mass function $\mathbb{P}$ describes probabilities of instantiations, i.e., a function $\mathbb{P} : \mathrm{Inst}(\mathbf{X}) \to [0, 1]$ and is a joint distribution over $\mathbf{X}$ if $\sum_{\mathbf{x} \in \mathrm{Inst}(\mathbf{X})} \mathbb{P}(\mathbf{x}) = 1$
- Evidence is a partial instantiation, an assignment to only some variables of $\mathbf{X}$

Cloudy	**R**ain	**S**prinkler	**W**et	$\mathbb{P}(C, R, S, W)$
0	0	0	0	0.225
0	0	0	1	0
0	0	1	0	0.045
0	0	1	1	0.18
0	1	0	0	0.0025
0	1	0	1	0.0225
0	1	1	0	0
0	1	1	1	0.025
1	0	0	0	0.09
1	0	0	1	0
1	0	1	0	0.002
1	0	1	1	0.008
1	1	0	0	0.036
1	1	0	1	0.324
1	1	1	0	0
1	1	1	1	0.04

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1	1	0	0	0.036
1	1	0	1	0.324
1	1	1	0	0
1	1	1	1	0.04

- probability of instantiation **x**

$$\mathbb{P}(C = 0, R = 1, S = 0, W = 1) = 0.0225$$
$$\mathbb{P}(C_0, R_1, S_0, W_1) = 0.0225$$

Probabilistic Inference

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- probability of instantiation **x**

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$$\mathbb{P}(C_0,R_1,S_0,W_1) = 0.0225$$

- probability of evidence e

$$\mathbb{P}(S_0,W_1) = 0+0.0225+0+0.324$$
$$= 0.3465$$

Probabilistic Inference

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0	1	0	1	0.0225
0	1	1	0	0
0	1	1	1	0.025
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- probability of instantiation $\mathbf{x}$

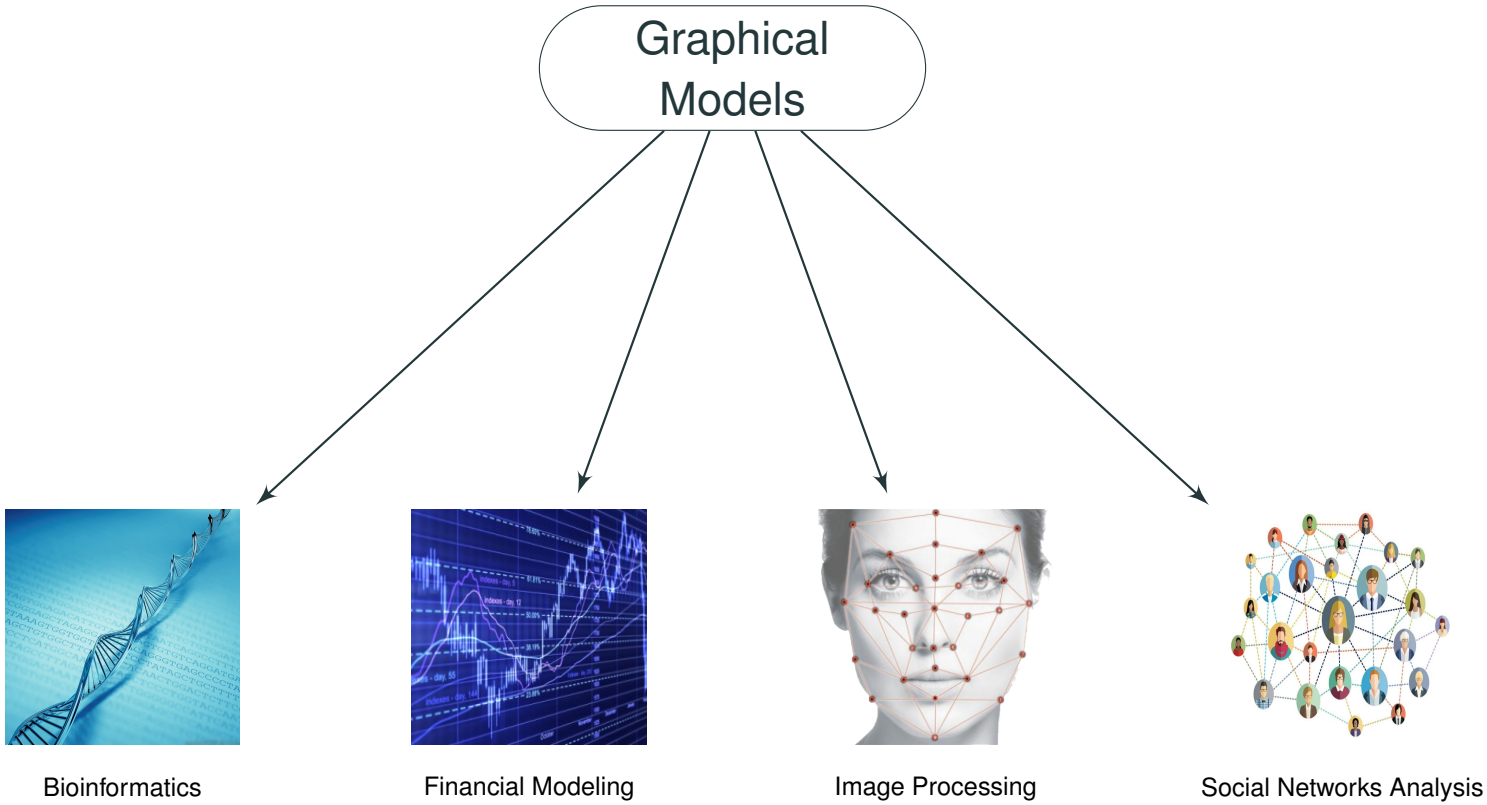
$$\mathbb{P}(C = 0, R = 1, S = 0, W = 1) = 0.0225$$
$$\mathbb{P}(C_0, R_1, S_0, W_1) = 0.0225$$

- probability of evidence e

$$\mathbb{P}(S_0, W_1) = 0 + 0.0225 + 0 + 0.324$$
$$= 0.3465$$

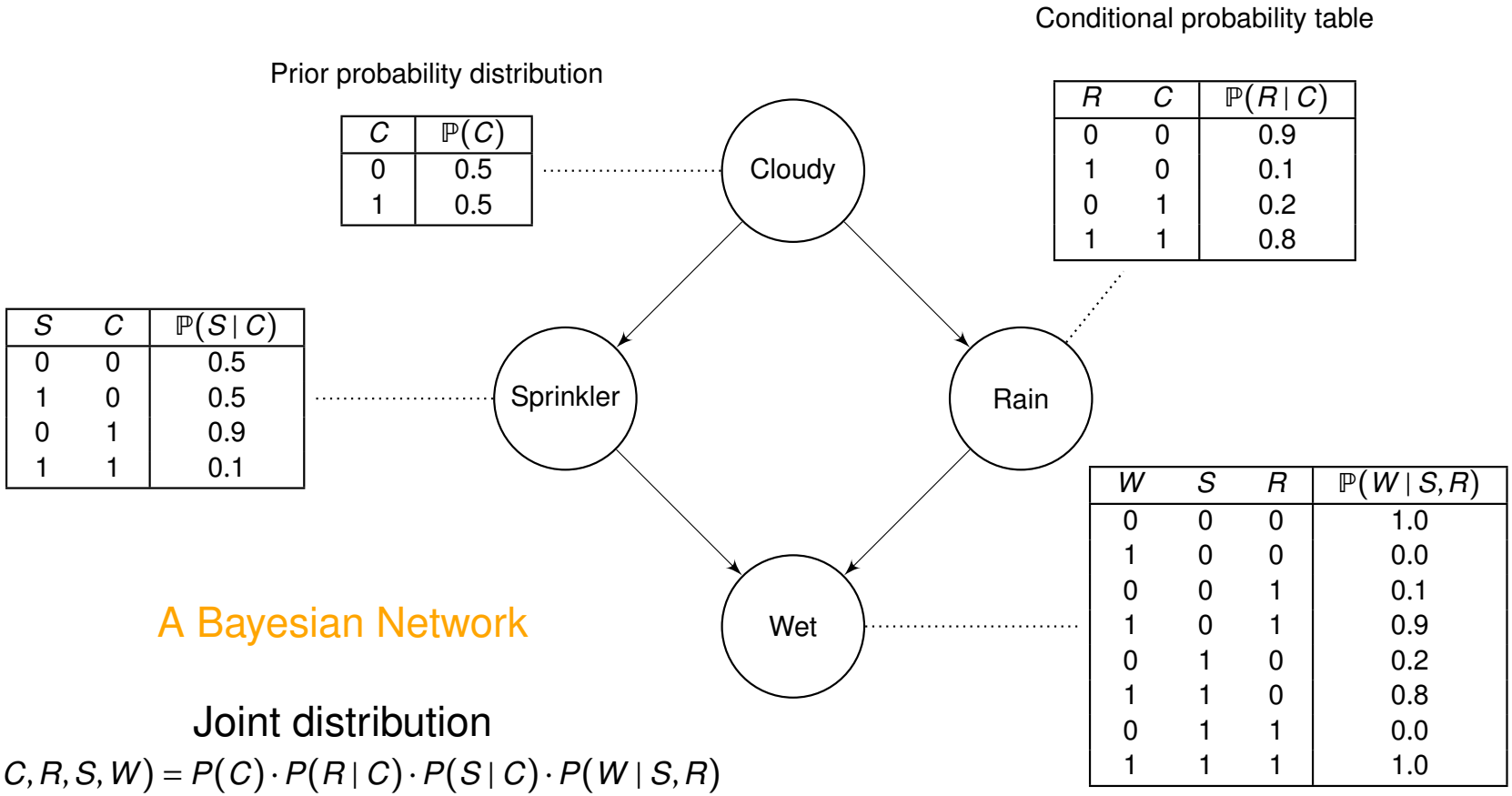
- The size of the joint distribution is exponential in the number of variables!

Graphical Models



Graphical models are a well-studied framework for representing high-dimensional probability distributions, with a wide spectrum of applications.

Graphical Models



C	$\mathbb{P}(C)$
0	0.5
1	0.5

R	C	$\mathbb{P}(R \mid C)$
0	0	0.9
1	0	0.1
0	1	0.2
1	1	0.8

S	C	$\mathbb{P}(S \mid C)$
0	0	0.5
1	0	0.5
0	1	0.9
1	1	0.1

W	S	R	$\mathbb{P}(W \mid S,R)$
0	0	0	1.0
1	0	0	0.0
0	0	1	0.1
1	0	1	0.9
0	1	0	0.2
1	1	0	0.8
0	1	1	0.0
1	1	1	1.0

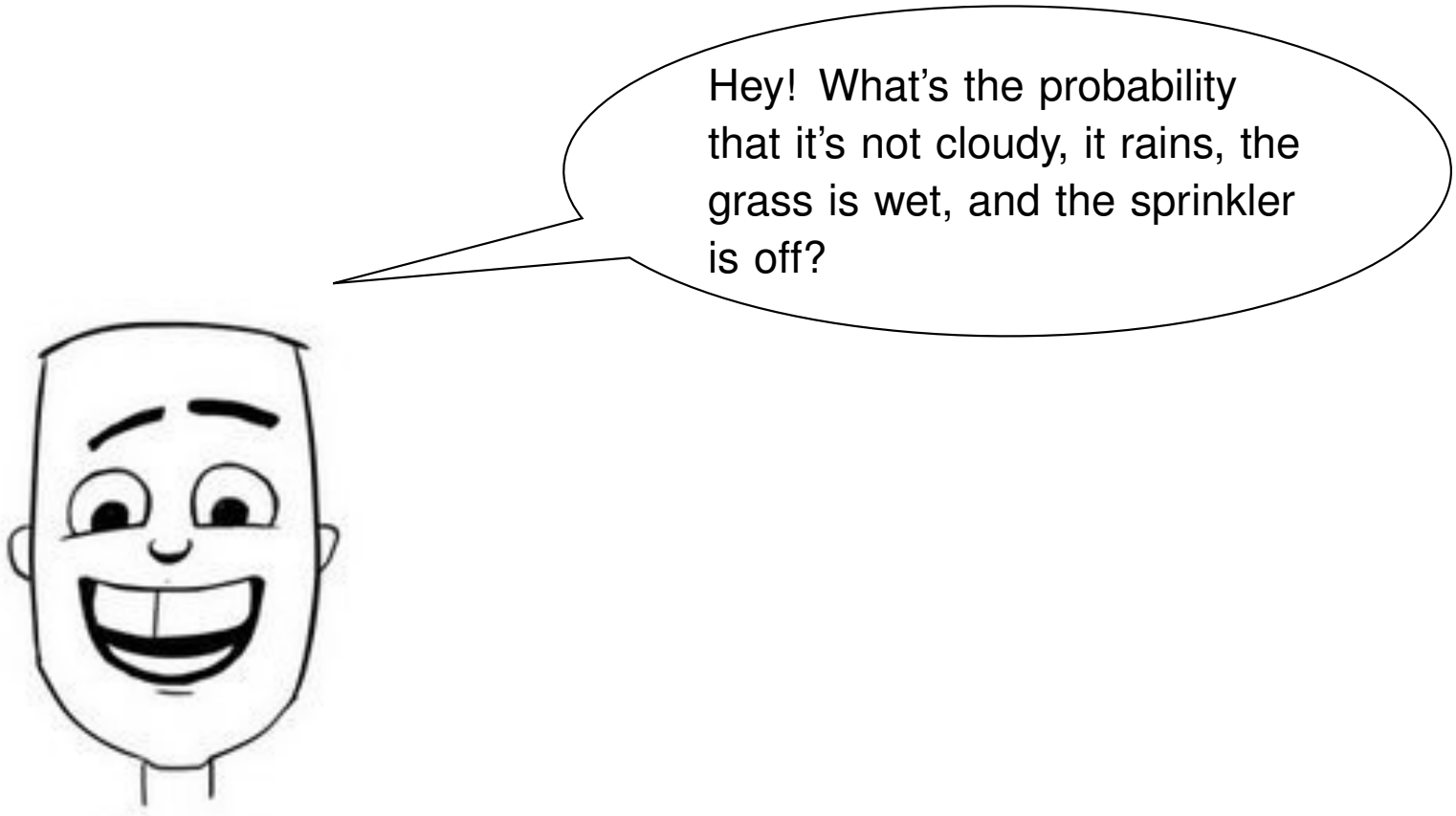
Joint distribution

$$P(C,R,S,W) = P(C) \cdot P(R \mid C) \cdot P(S \mid C) \cdot P(W \mid S,R)$$

Graphical models are commonly separated into:

- directed models (a.k.a Bayesian networks), with conditional probability tables (CPTs) over a directed (acyclic) graph,
- undirected models (a.k.a. Markov networks), with energy functions over the cliques of an undirected graph.

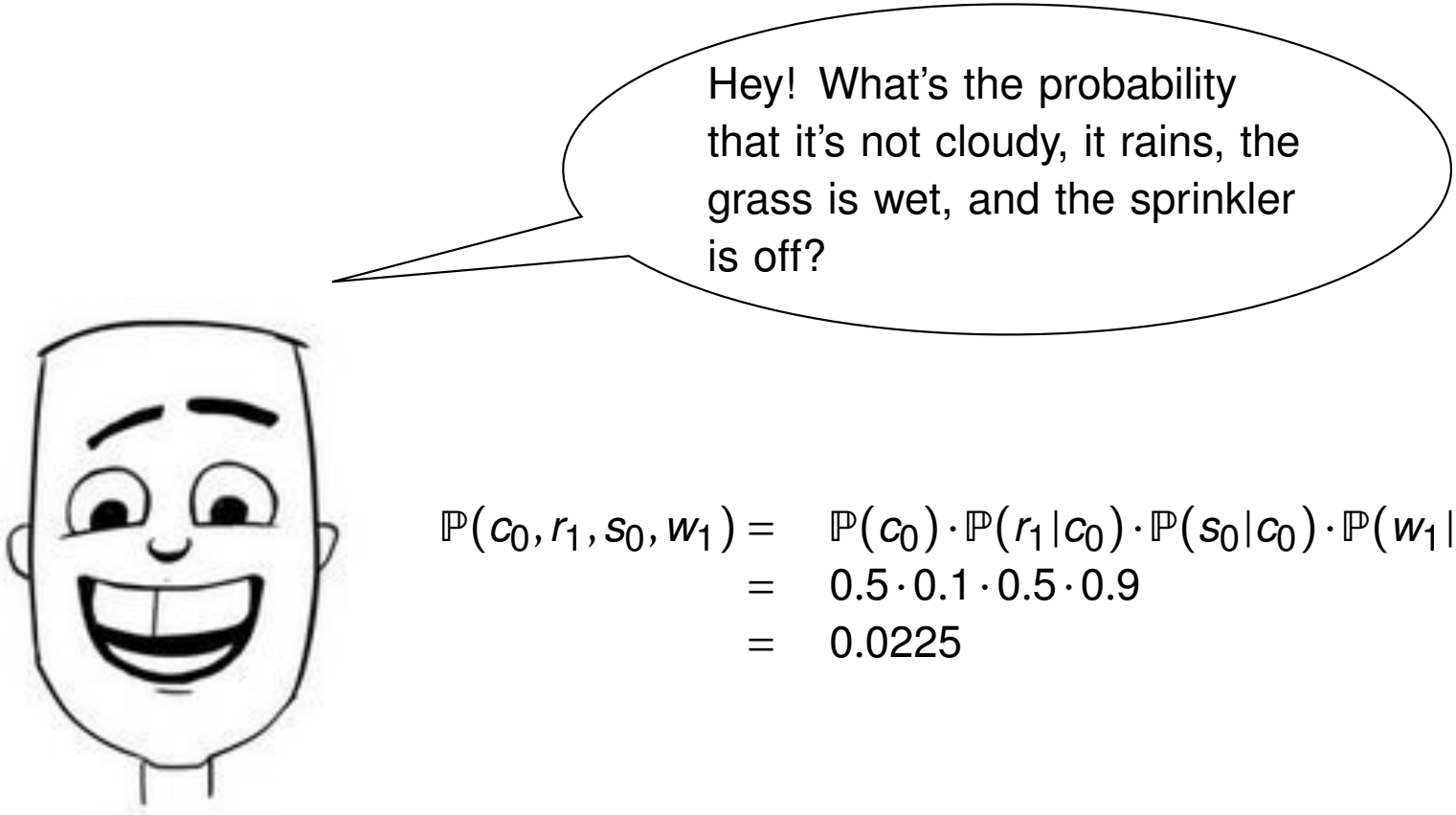
Bayesian network



The semantics of Bayesian networks implies the following joint distribution:

$$\mathbb{P}(x_1, x_2, \ldots, x_n) = \prod_i \mathbb{P}(x_i | u_i)$$

Bayesian network

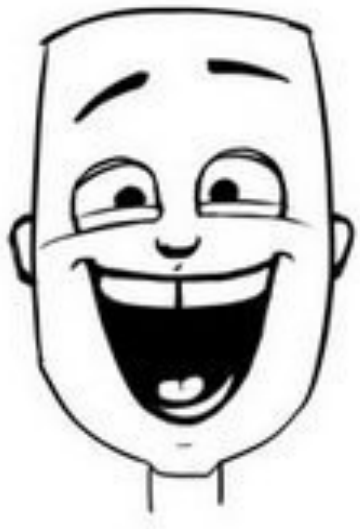


$$\begin{aligned}\mathbb{P}(c_0, r_1, s_0, w_1) &= \mathbb{P}(c_0) \cdot \mathbb{P}(r_1|c_0) \cdot \mathbb{P}(s_0|c_0) \cdot \mathbb{P}(w_1|s_0, r_1) \\ &= 0.5 \cdot 0.1 \cdot 0.5 \cdot 0.9 \\ &= 0.0225\end{aligned}$$

The semantics of Bayesian networks implies the following joint distribution:

$$\mathbb{P}(x_1, x_2, \ldots, x_n) = \prod_i \mathbb{P}(x_i|u_i)$$

Probabilistic Inference



Product rule: $\mathbb{P}(A, X) = \mathbb{P}(A|X) \cdot \mathbb{P}(X)$

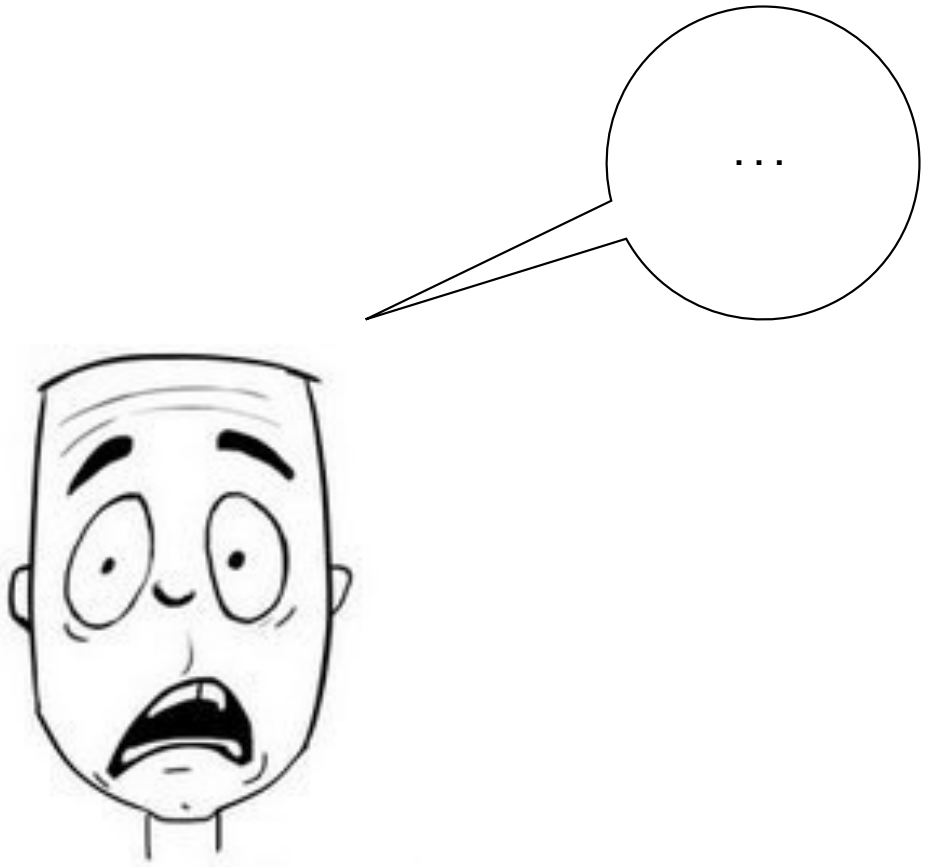
$$\mathbb{P}(w_1 \mid c_1) = \frac{\mathbb{P}(w_1, c_1)}{\mathbb{P}(c_1)}$$

Bayes' rule: $\mathbb{P}(A|X) = \frac{\mathbb{P}(X|A) \cdot \mathbb{P}(A)}{\mathbb{P}(X)}$

$$\mathbb{P}(c_1 \mid s_1) = \frac{\mathbb{P}(s_1|c_1) \cdot \mathbb{P}(c_1)}{\mathbb{P}(s_1)}$$

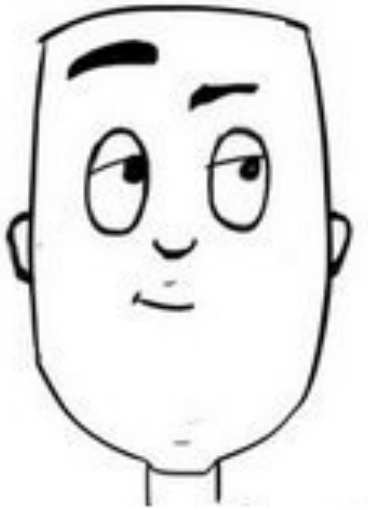
Probabilistic inference is a basic task for handling more complex queries:

- Conditional probabilities: evaluating the probability of an event $\boldsymbol{y}$ given an evidence $\boldsymbol{x}$;
- Most probable explanations: given an evidence $\boldsymbol{x}$, find an assignment $\boldsymbol{y}$ over the complementary variables that maximizes $\mathbb{P}(\boldsymbol{y} \mid \boldsymbol{x})$;
- ...



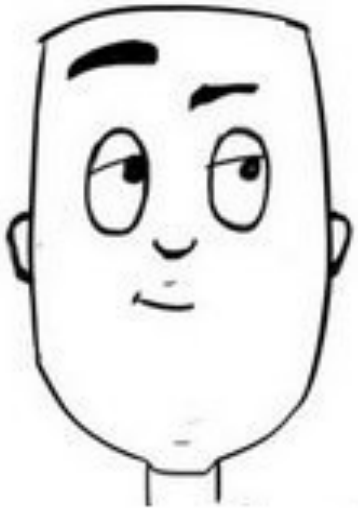
Unfortunately, probabilistic inference is #**P-complete** in graphical models, including both Bayesian networks and Markov networks.

Probabilistic Inference to Weighted Model Counting



So . . . Can we **reduce** the problem of probabilistic inference to Weighted Model Counting?

Probabilistic Inference to Weighted Model Counting



So . . . Can we **reduce** the problem of probabilistic inference to Weighted Model Counting?

Yes! The idea is to find a translation function τ mapping

- any graphical model (Bayes or Markov net) N to a weighted CNF formula $\tau(N)$, and
- any partial assignment $\boldsymbol{x}$ to a term $\tau(\boldsymbol{x})$,

such that

$$\mathbb{P}_N(\boldsymbol{x}) = \frac{W[\tau(N) \wedge \tau(x)]}{W[\tau(N)]}$$

where $W[\tau(N)]$ is the **partition constant** ($= 1$ for BNs).